CS171: Cryptography

Lecture 14

Sanjam Garg

Cryptographic Group

- If p and q are primes such that $2q \ = \ p - 1$ and let g ∈ Z_p^* be an elements of order q. Let $H \ = \langle g \rangle$ be the group of order q.
- Example, p = 23 and q = 11
- $Z_p^* = \{1, 2, \ldots 22\}$ and $a \cdot b = ab \; mod \; 23$

$\langle g \rangle$

- $Z_p^* = \{1, 2, \ldots 22\}$
- $\langle 1 \rangle = \{1\}$
- $\langle 2 \rangle = \{2, 4, 8, 16, 9, 18, 13, 3, 6, 12, 2^{11} = 1\}$
- $\langle 5 \rangle = \{5, 2, 10, 4, 20, 8, 17, 16, 11, 12, \ldots 5^{22} = 1\}$
- $\langle 22 \rangle = \{22, 22^2 = 1\}$

- Pick any g such that $g^{11} = 1$.
- For example, $\mathrm{H} = \langle 2 \rangle$ is of prime order
- For hardness use large primes.

The Discrete-Log Problem

- Let $\mathcal{G}(1^n)$ be a PPT algorithm that generates description of a cyclic group, i.e., order q (where $|q| = n$) and a generator g.
- Unique bit representation for each element and group operation can be performed in time polynomial in n.
- Sampling a uniform group element: Sample $x \leftarrow Z_q$ and compute g^x.

DLOG Problem

$\mathrm{DLog}_{A,\mathcal{G}}$(n)

1. Run $\mathcal{G}(1^n)$ to obtain (G, g, q).
2. Pick uniform $h \in G$.
3. A is given (G, g, q, h) and it outputs x.
4. Output 1 if $g^x = h$ and 0 otherwise

Discrete-Log Problem is hard relative to $\mathcal{G}$ if

$\forall$ $PPT\ A$ $\exists$ $negl$ such that:

$\left|\Pr\left[\mathrm{DLog}_{A,\mathcal{G}}(n) = 1\right]\right| \leq$ negl(n).

The Diffie-Hellman Problems

- The computational variant: given g^x and g^y compute g^{xy}
- The decisional variant: given g^x and g^y distinguish between g^{xy} and a random group element.

Computational Diffie-Hellman Problem

$\mathrm{CDH}_{A,\mathcal{G}}(n)$

1. Run $\mathcal{G}(1^n)$ to obtain (G, g, q).
2. $a, b \leftarrow Z_q^*$.
3. A is given (G, g, q, g^a, g^b) and it outputs h.
4. Output 1 if $g^{ab} = h$ and 0 otherwise

CDH is hard relative to $\mathcal{G}$ if

$\forall$ $PPT\ A$ $\exists$ $negl$ such that:

$\left|\Pr\left[\mathrm{CDH}_{A,\mathcal{G}}(n) = 1\right]\right| \leq \mathrm{negl}(n).$

Decisional Diffie-Hellman Problem

$\mathrm{DDH}_{A,\mathcal{G}}$(n)

1. Run $\mathcal{G}(1^n)$ to obtain (G, g, q).
2. $a, b, r \leftarrow Z_q^*$. Sample a uniform bit c.
3. A is given $(G, g, q, g^a, g^b, g^{ab+cr})$ and it outputs c'.
4. Output 1 if $c = c'$ and 0 otherwise

DDH is hard relative to $\mathcal{G}$ if

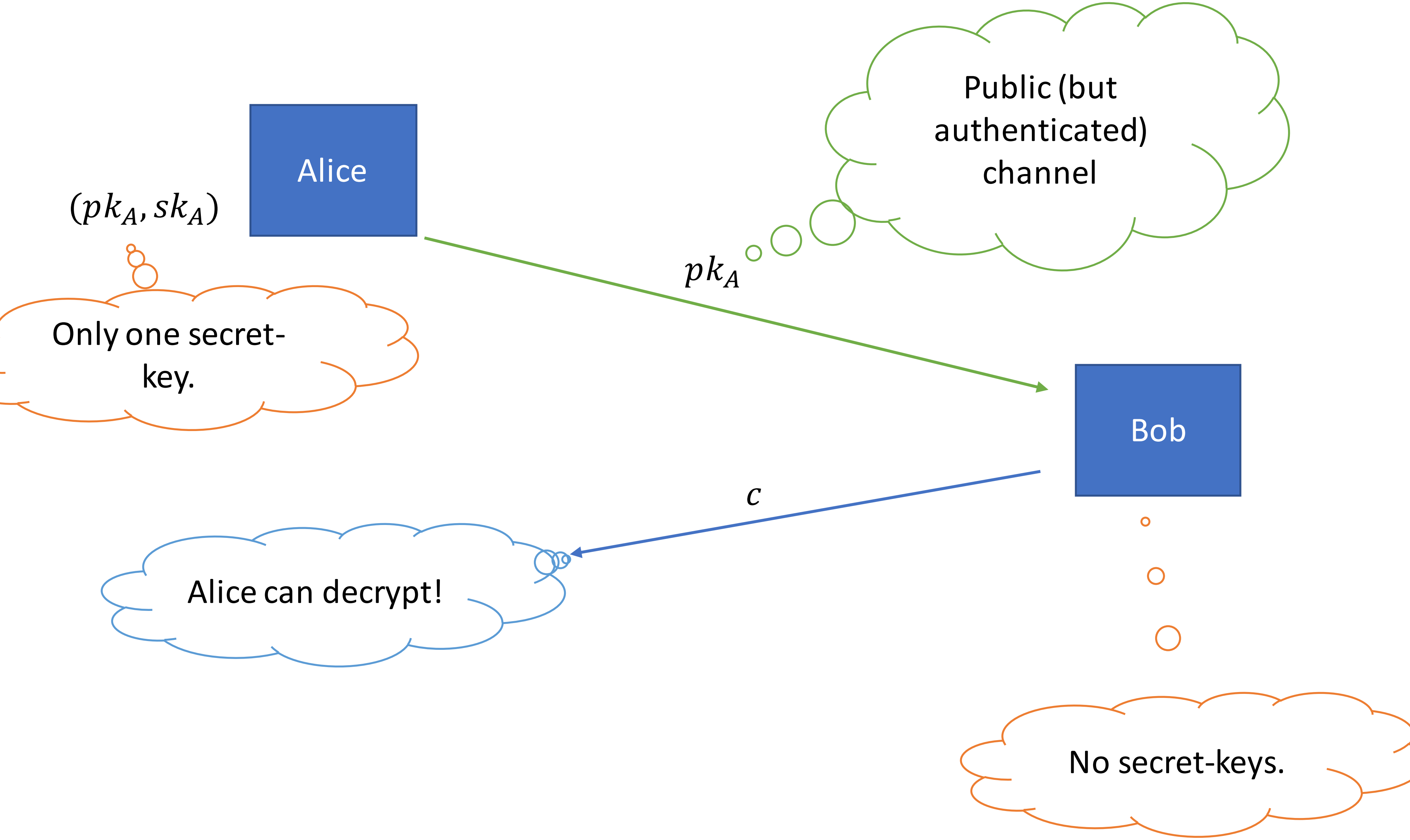
$\forall$ $PPT\ A$ $\exists$ $negl$ such that:

$\left|\Pr\left[\mathrm{DDH}_{A,\mathcal{G}}(n) = 1\right]\right| \leq$ ½ + negl(n).

Public-Key Cryptography

- Public-Key Encryption
- Digital Signatures

Public-Key Encryption



Public-Key Encryption vs Private-Key Encryption

- Public-key encryption is strictly stronger than private-key encryption

- Then why even use private-key encryption?
  - Public-key encryption is roughly 2-3 orders of magnitude slower than private-key encryption

Public-Key Encryption

- A public-key encryption scheme is a triple of PPT algorithms (Gen, Enc, Dec) such that:

1. $Gen(1^n) \to (pk, sk)$
2. $Enc(pk, m) \to c$
3. $Dec(sk, c) \to m/\bot$

- Correctness: For all (pk, sk) output by $Gen(1^n)$, we have that $\forall$ (legal) $m, Dec\left(sk, Enc(pk, m)\right) = m$
- Security: EAV-security, CPA-security?

EAV Security

$\mathrm{PubK}^{\mathrm{eav}}_{A,\Pi}(n)$

1. $(pk, sk) \leftarrow G(1^n)$ and give pk to A.
2. A outputs $m_0, m_1 \in \{0,1\}^*, |m_0| = |m_1|$.
3. b $\leftarrow$ {0,1}, $c \leftarrow Enc(pk, m_b)$
4. c is given to A and it outputs b'
5. Output 1 if $b = b'$ and 0 otherwise

Encryption scheme $\Pi = (Gen, Enc, Dec)$ is indistinguishable in the presence of an eavesdropper, or is *EAV-secure* if

$\forall$ PPT A it holds that:

$$\Pr\left[\mathrm{PubK}^{\mathrm{eav}}_{A,\Pi} = 1\right] \leq \frac{1}{2}$$

+ negl(n)

EAV-security vs CPA Security

- In the public-key setting the two notions are identical.
- Since, given the public-key, encryption can be performed (without any secret values)
- Hence, encryption must be randomized

What about security of multiple messages?

- CPA-security implies security for encrypting multiple messages (same as the private-key setting)
- $Enc(pk, m_1 \ldots m_n)$: $Enc(pk, m_1) \ldots Enc(pk, m_n)$
- Proof via a direct hybrid argument

CCA Security (A bigger concern in the PKE setting)

- Attacker can obtain decryptions of ciphertexts of its choice itself

- Attacker can more easily come up with illegitimate ciphertexts (cannot have a MAC on a ciphertext)

- Malleability: An attacker can given a ciphertext c encrypting a message m could obtain a ciphertext c' of a related message m' (without knowing m' itself)

CCA Security

Much harder in the PKE setting.

$\text{PubK}^{\text{CCA}}_{\text{A},\Pi}(n)$

1. $(pk, sk) \leftarrow G(1^n)$ and give pk to A.
2. $A^{Dec(sk,\cdot)}$ outputs $m_0, m_1 \in \{0,1\}^*, |m_0| = |m_1|$.
3. b ← {0,1}, $c^* \leftarrow Enc(pk, m_b)$
4. c is given to $A^{Dec(sk,\cdot)}$ and it outputs b' (query c^* not allowed)
5. Output 1 if $b = b'$ and 0 otherwise

Encryption scheme $\Pi = (Gen, Enc, Dec)$ is indistinguishable in the presence of a CCA attacker, or is *CCA-secure* if

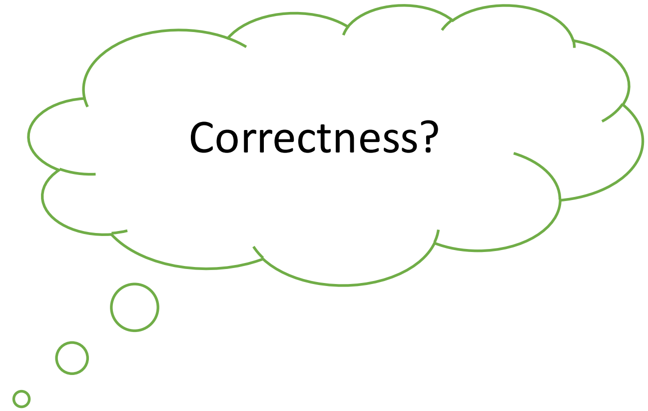
$\forall$ PPT A it holds that:

$$\Pr\left[\text{PubK}^{\text{cca}}_{\text{A},\Pi} = 1\right] \le \frac{1}{2}$$

+ negl(n)

Construction of PKE

ElGamal Encryption



1. $Gen(1^n) \to (pk, sk)$
   1. Run $\mathcal{G}(1^n)$ to obtain (G, g, q).
   2. Sample $x \leftarrow Z_q$ and set $h = g^x$
   3. Set $pk = (G, g, q, h)$ and $sk = x$.
2. $Enc(pk, m \in G) \to c = (c_1, c_2)$
   1. Parse $pk = (G, g, q, h)$
   2. Sample $r \leftarrow Z_q$ and set $c_1 = g^r$ and $c_2 = m \cdot h^r$
3. $Dec(sk, c) \to m/\bot$
   1. Parse $c = (c_1, c_2)$
   2. Output $\frac{c_2}{c_1^r}$

Encrypting long messages

- Encrypting block-by-block is inefficient
  - Ciphertext expands for each block
  - Public-key encryption is “expensive”

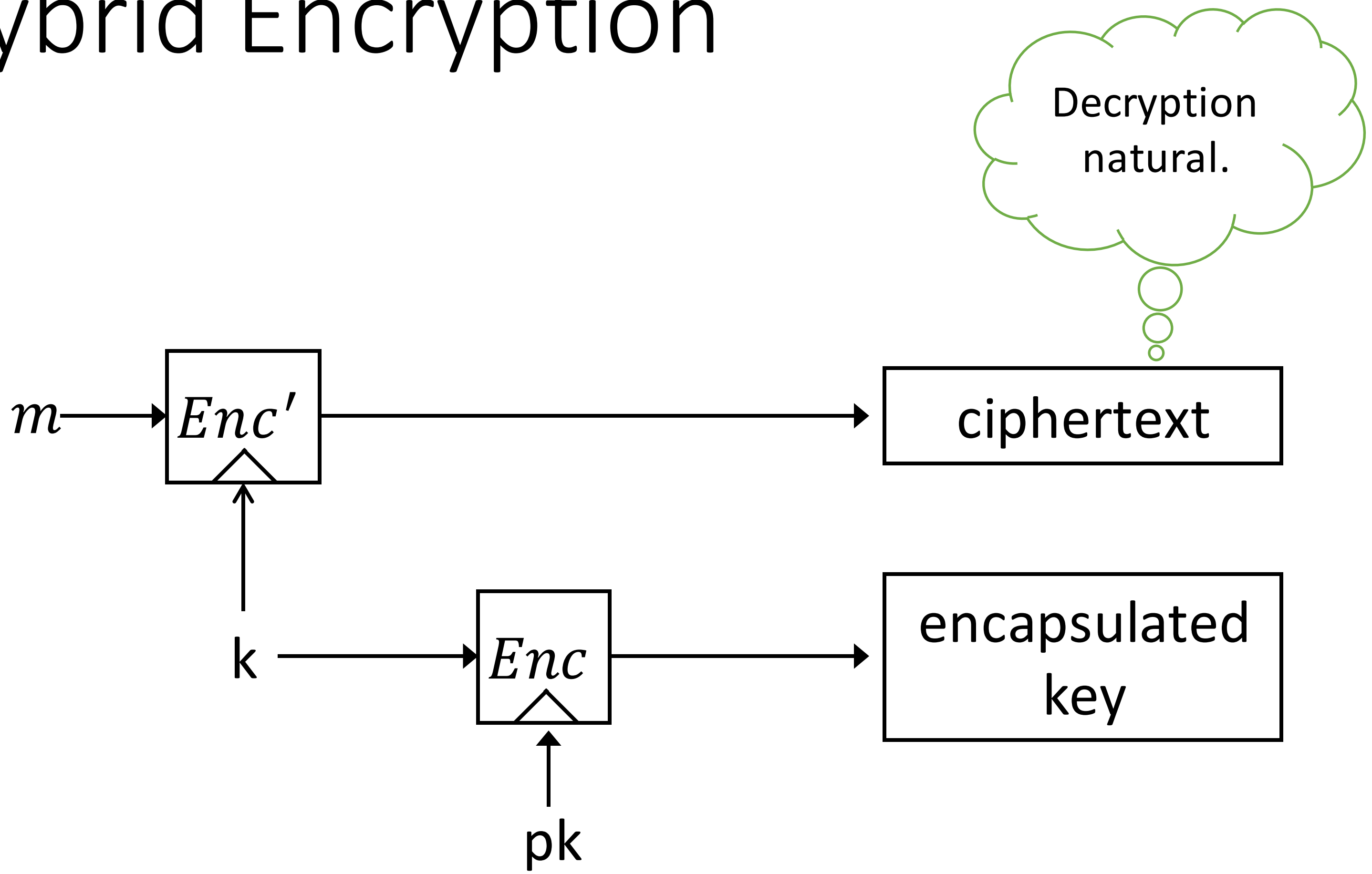
- Anything better?

Hybrid Encryption

- Use public-key encryption to set up a shared secret-key k which is then used to encrypt the message itself

- Benefits:
  - The inefficiency of the public-key encryption is not the bottleneck; i.e. we get amortized efficiency as the message is large
  - The ciphertext expansion over the message is small

Hybrid Encryption



The *functionality* of public-key encryption
at the (asymptotic) *efficiency* of private-key encryption!

Thank You!