CS171: Cryptography

Lecture 4

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Defining Computationally Secure Encryption (syntax)

- A *private-key encryption scheme* is a tuple of algorithms (Gen, Enc, Dec):
  - $Gen(1^n)$: outputs a key k (assume $|k| > n$)
  - Enc_k(m): takes key k and message $m \in \{0,1\}^*$ as input; outputs ciphertext c
$$c \leftarrow Enc_k(m)$$
  - Dec_k (c): takes key k and ciphertext c as input; outputs m or "error"
$$m := Deck(c)$$

Correctness: For all n, k output by $Gen(1^n)$, $m \in \{0,1\}^*$ it holds that $Dec_k(Enc_k(m)) = m$

Computational Indistinguishability

$\mathrm{PrivK}^{\mathrm{eav}}_{A,\Pi}$ (n)

1. A outputs $m_0, m_1 \in$ ~~$\mathcal{M}$.~~ $\{0,1\}^*, |m_0| = |m_1|$
2. $b \leftarrow \{0,1\}$, $k \leftarrow Gen(1^n)$, $c \leftarrow Enc_k(m_b)$
3. c is given to A
4. A output b'
5. Output 1 if $b = b'$ and 0 otherwise

Encryption scheme $\Pi = (Gen, Enc, Dec)$ with ~~message space $\mathcal{M}$~~ is ~~perfectly~~ computationally indistinguishable if $\forall$ PPT A it holds that:

$$\Pr\left[\mathrm{PrivK}^{\mathrm{eav}}_{A,\Pi}(n) = 1\right] \leq \frac{1}{2}$$

+ negl(n)

Does not hide message length! A scheme that only supports messages of fixed length is called a fixed-length encryption scheme.

Constructing Secure Encryption

Pseudorandom Generators (a building block)

What does it mean to be random?

- Is this string random?
  - 01010101010101010101
  - 01010001011010101010

- Uniformity is a property of a *distribution* and not a specific *string*.
- A distribution on *n*-bit strings is a function $D\colon \{0,1\}^n \to [0,1]$ such that $\Sigma_x D(x) = 1$
  - For *uniform* distribution on *n*-bit strings, denoted U_n, $\forall x \in \{0,1\}^n$ we set $D(x) = 2^{-n}$

What about pseudorandomness?

- Intuitively: should be indistinguishable from uniform.

- As before: pseudorandomness is a property of a *distribution* and not a specific *string*

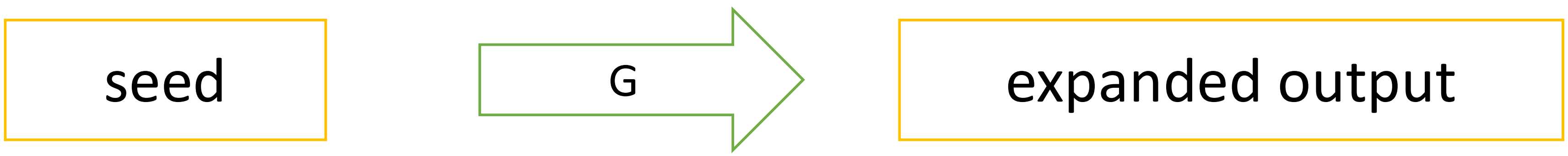
Pseudorandom Generators PRG

- Stretches a short uniform ``seed’’ into a larger ``uniform looking’’ larger output

- Useful when only a few random bits are available.

Pseudorandom Generators

- G: $\{0,1\}^n \rightarrow \{0,1\}^{\ell(n)}$, where $\ell(n) > n$



- G is pseudorandom generator if $\forall$ PPT A we have $\exists$ $negl(\cdot)$ such that,
$$\left| \Pr_{x \leftarrow U_{\ell(n)}}[A(x) = 1] - \Pr_{s \leftarrow U_n}[A(G(s)) = 1] \right| \leq negl(n)$$

PRG (Predicting Game Style)

$\mathrm{PRG}_{\mathrm{A,G}}(1^n)$

1. b ← {0,1},
2. If $b = 0$ set $x \leftarrow G(U_n)$ else set $x \leftarrow U_{\ell(n)}$.
3. Give x to A
4. A output b'
5. Output 1 if $b = b$' and 0 otherwise

G is a PRG if

$\forall$ $PPT\ A$ it holds that:

$$\Pr\left[\mathrm{PRG}_{\mathrm{A,G}}(1^n) = 1\right] \leq \frac{1}{2} + negl(n)$$

Seed must be kept secret. Analogous to the secret key in an encryption scheme.

Fixed-Length Encryption Scheme

Let G be a PRG: $\{0,1\}^n \rightarrow \{0,1\}^{\ell(n)}$.

- $Gen(1^n)$: Choose uniform $k \in \{0,1\}^n$ and output it as the key
- $Enc_k(m)$: On input a message $m \in \{0,1\}^{\ell(n)}$ output the ciphertext
$$c \coloneqq G(k) \oplus m$$
- $\mathrm{Dec}_k(c)$: On input a ciphertext $c \in \{0,1\}^{\ell(n)}$ output the message
$$m \coloneqq G(k) \oplus c$$

Proof of Security

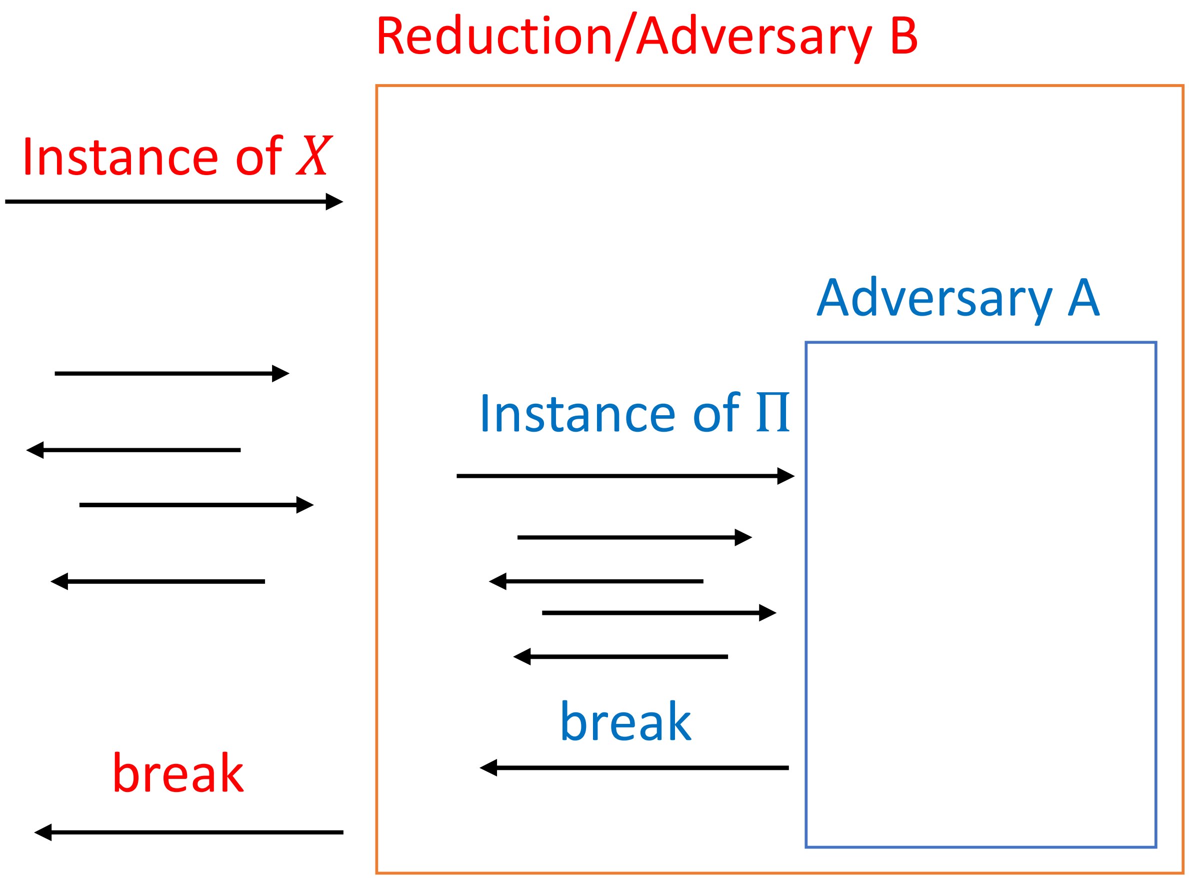
Theorem: If G is a PRG, then this construction is a fixed-length private-key encryption scheme that has indistinguishable encryptions in the presence of an eavesdropper.

Proof by Reduction (If X then Π)

- To Prove: If no PPT B breaks X, then no PPT A breaks Π

- Assume there exists a PPT A that ``breaks'' $\Pi$, then we construct PPT $B$ that ``breaks'' X
- However, such a B cannot exist. Thus, our assumption that there exists A that ``breaks'' Π must have been false.

Proof by Reduction (If X then Π)



Important:

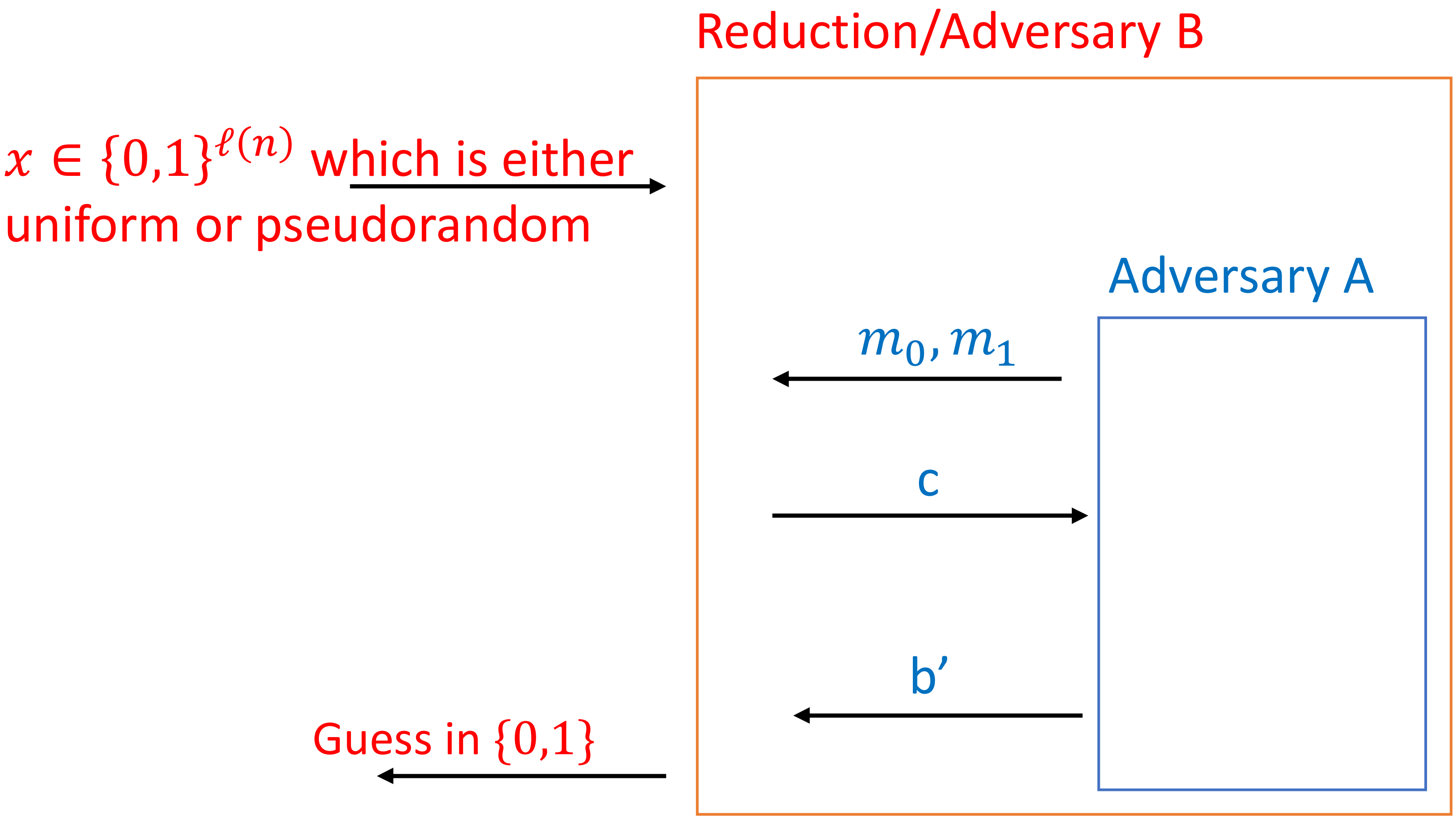
1. View of A: No change
2. B is PPT given A is PPT
3. B succeeds with degrades wrt. A's by 1/poly(n)

Proof of Security

Theorem: If G is a PRG, then this construction is a fixed-length private-key encryption scheme that has indistinguishable encryptions in the presence of an eavesdropper.

- Proof by reduction: Given a PPT adversary A ``breaking'' the encryption scheme construct a PPT adversary B ``breaking'' the PRG

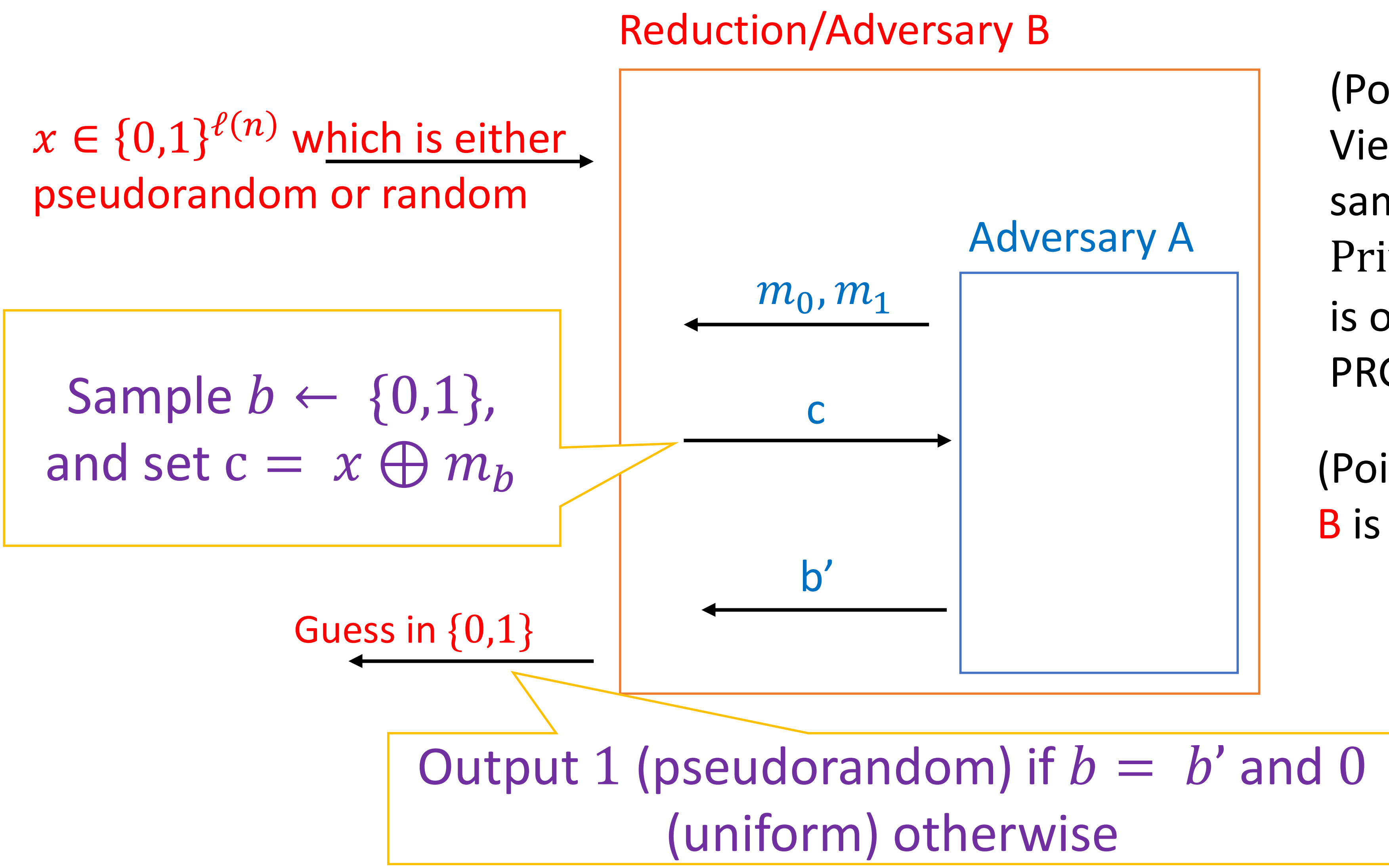
Proof by Reduction (If PRG then Indistinguishable Encryption)



Given: $\Pr[\mathrm{PrivK}^{\mathrm{eav}}_{\mathrm{A},\Pi}(n) = 1] \geq ½ + \epsilon(n)$

To prove: $|\Pr[B(G(U_n)) = 1] - \Pr[B(U_{\ell(n)}) = 1]| \geq \delta(n)$ or $\Pr[\mathrm{PRG}_{B,G}(1^n) = 1] \geq \frac{1}{2} + \delta'(n)$

Proof by Reduction (If PRG then Indistinguishable Encryption)



(Point 1): View of A is same as in $\mathrm{PrivK}^{\mathrm{eav}}_{\mathrm{A},\Pi}$ if x is output of PRG

(Point 2): B is PPT

(Point 3) Success of B

1. If x is sampled from $U_{\ell(n)}$, then $\Pr[b = b'] = \frac{1}{2}$.
   - The scheme behaves like a one-time pad.
2. If x is sampled from $G(U_n)$, then $\Pr[b = b'] \geq$ ½ + ϵ(n)
3. $\Pr[B\ guesses\ correct] =$
   $.5\Pr[B\ guesses\ correct|\ x$ is from $U_{\ell(n)}] +$
   $.5\Pr[B\ guesses\ correct|\ x$ is from $G(U_n)]$

$$= \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2} + \epsilon(n)\right)$$

$$= \frac{1}{2} + \frac{\epsilon(n)}{2}$$

Lessons

- Pseudo OTP is secure
  - Assuming G is a PRG
  - With respect to our definition
- Gain: Pseudo OTP has a short key
  - n bits instead of $\ell(n)$ bits
- Does pseudo OTP allow encryption of multiple messages?
  - Let's first define it!

Practice Question

Step 2: Prove H is not a PRG!

- Let $G:\{0,1\}^n \rightarrow \{0,1\}^{2n}$ be a PRG, then is $H:\{0,1\}^n \rightarrow \{0,1\}^{2n}$ a PRG?

$$H(s) = (s||0^n) \oplus G(s)$$

For any G!

- Yes?
- No?

Step 1: Prove G is a PRG!

- No! Let $F:\{0,1\}^{n/2} \rightarrow \{0,1\}^{3n/2}$ be a PRG then

$G(s = (s_0, s_1)) = s_0||F(s_1)$, where $s_0, s_1 \in \{0,1\}^{n/2}$

Step 1: G is a PRG

- Given: F is a PRG
- To Prove: $G(s = (s_0, s_1)) = s_0 || F(s_1)$ is a PRG
- Proof:

1. Assume G is not a PRG
2. $\exists A$, such that $\left| \Pr_{x \leftarrow U_{2n}}[A(x) = 1] - \Pr_{s \leftarrow U_n}\left[A\left(G(s)\right) = 1\right] \right| \geq \epsilon(n)$
3. $\exists A$, such that $\left| \Pr_{x \leftarrow U_{2n}}[A(x) = 1] - \Pr_{s_0 \leftarrow U_{\frac{n}{2}},\, s_1 \leftarrow U_{\frac{n}{2}}}[A(s_0 || F(s_1)\,) = 1] \right| \geq \epsilon(n)$
4. $\exists B$, such that $\left| \Pr_{x \leftarrow U_{3n/2}}[B(x) = 1] - \Pr_{s_1 \leftarrow U_{\frac{n}{2}}}[B(F(s_1)\,) = 1] \right| \geq \epsilon(n)$
5. F is not a PRG, contradicting the given. Thus, G must be a PRG.

Step 2: H is not a PRG

$$H(s) = (s||0^n) \oplus G(s)$$
$$= (s_0||s_1||0^n) \oplus (s_0||F(s_1))$$
$$= 0^{\frac{n}{2}}||((s_1||0^n) \oplus F(s_1))$$

Thank You!